1. **Solution (a):** For $i = 1, 2, .., 5$, let A_i be the event that path i is the only closed path. A_i's are disjoint events, $P(A_i) = p^4(1-p)$, for all i. Then $B = \bigcup_{i=1}^{5} A_i$ is the event that exactly four paths are open. Let C be the event that water flows to the layer of sandstone. $P(C|A_i) = 1$, for $i = 1, 2, 3, 4$ and $P(C|A_5) = 0$. Then

$$\begin{aligned} P(C|B) &= \frac{P(C \cap B)}{P(B)} \\ &= \frac{\sum_{i=1}^{5} P(C|A_i)P(A_i)}{\sum_{i=1}^{5} P(A_i)} = \frac{4}{5} \end{aligned}$$

Solution (b): Let D_i be the event that path i is open. Then $C = (D_5 \cap D_3 \cap D_1) \cup (D_5 \cap D_4 \cap D_2)$. Therefore, by independence of D_i's and sum rule, $P(C) = 2p^3 - p^5$

$$\begin{aligned} P(B|C) &= \frac{P(C|B)P(B)}{P(C)} \\ &= \frac{\frac{4}{5} * 5p^4(1-p)}{2p^3 - p^5} \\ &= \frac{4p(1-p)}{2-p^2} \end{aligned}$$

□

2. **Solution:** Polya's Urn Scheme (with b black balls and r red balls, and adding c balls at each stage). R_i is the event that the i^{th} draw yields a red ball. Then

$$\begin{aligned} P(R_2|R_3) &= \frac{P(R_2 \cap R_3)}{P(R_3)} \\ &= \frac{P(R_1 \cap R_2 \cap R_3) + P(R_1^c \cap R_2 \cap R_3)}{P(R_1 \cap R_2 \cap R_3) + P(R_1^c \cap R_2 \cap R_3) + P(R_1 \cap R_2^c \cap R_3) + P(R_1^c \cap R_2^c \cap R_3)} \\ &= \frac{r(r+c)(r+2c) + b(r)(r+c)}{r(r+c)(r+2c) + b(r)(r+c) + r(b)(r+c) + b(b+c)(r)} \end{aligned}$$

□

3. **Solution:** Let N be the number of empty poles when r flags of different colours are displayed randomly on n poles arranged in a row (here $r, n \in \mathbb{N}$ with $r \geq n$). Assume that there is no limitation on the number of flags on each pole. Let r flags be chosen randomly to be put on randomly chosen poles. If we follow this procedure, observe that the elementary outcomes have different probabilities. Let $A_{i,k}$ be the event that pole i is not chosen for flag k. $A_i = \cap_{k=1}^{r} A_{i,k}$ is the event that pole i is empty. $B = (\cup_{i=1}^{n} A_i)^c$ is the event that no pole is empty. By independence, $P(A_i) = \prod_{k=1}^{r} P(A_{i,k}) = (\frac{n-1}{n})^r$. Similary $P(\cap_{m=1}^{j} A_{i_m}) = \prod_{k=1}^{r} P(\cap_{m=1}^{j} A_{i_m,k}) = (\frac{n-m}{n})^r$. Therefore,

$$P(B) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \left(\frac{n-m}{n}\right)^r$$

□